Existence results about the nonlinear Schrödinger-Poisson equations

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Nonlinear Schrödinger equations (1)

- For $N \geq 2$, consider the following equation

$$-\Delta u + u = |u|^{p-2}u, \quad \lim_{|x|\to\infty} u(x) = 0 \text{ in } \mathbb{R}^N \tag{1}$$

where $p \in (2, 2^*)$, $\quad 2^* = 2N/(N-2)$ if $N > 2$ and $2^* = \infty$ if $N = 2$.

- $\tilde{u} \in H^1(\mathbb{R}^N)$ is a solution $\Leftrightarrow$ $\tilde{u}$ is a critical point of the functional

$$I(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 - \frac{1}{p}|u|^p \, dx, \quad u \in H^1(\mathbb{R}^N). \tag{2}$$

Nonlinear Schrödinger equations (2)

- There is a positive radial solution and are infinitely many radial solutions of (1) (Strauss, 1977)
- There is no nontrivial solution for $p \geq 2^*$.
- The positive radial solution of (1) is unique (Kwong, 1989)
- Every positive solution of (1) is radially symmetric about some point in $\mathbb{R}^N$ (Gidas, Ni and Nirenberg, 1981)
- There exists a k-sign changing radial solution of (1) for given $k \in \mathbb{N}$ (Bartsch and Willem, 1993)

Nonlinear Schrödinger equations (3)

- Consider the Schrödinger equation with more general nonlinearity

$$-\Delta u + u = f(u), \quad \lim_{|x|\to\infty} u(x) = 0 \text{ in } \mathbb{R}^N \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions :

  - $(F1) \lim_{t\to 0^+} f(t)/t = 0$
  - $(F2) \limsup_{t\to\infty} f(t)/t^p < \infty$ for some $p \in (1, (N+2)/(N-2))$
  - $(F3) \frac{1}{2}T^2 < F(T)$ for some $T > 0$ where $F(t) = \int_0^t f(s)\, ds$

- We call the condition $(F1) \sim (F3)$ the Berestycki-Lions condition.

Nonlinear Schrödinger equations (4)

- There is a positive radial least energy solution of (3). If f is odd, there are infinitely many radial solutions (Berestycki and Lions, 1983)
- Uniqueness of the positive solution of (3) is not known.
- Any least energy solution of (3) is radially symmetric up to translation (Byeon, Jeanjean and Maris, 2009)
- The Berestycki-Lions condition is almost optimal for the existence.

Nonlinear Schrödinger-Poisson equations

- Consider the following system of equations (NSP system)

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-2}u, \\ -\Delta\phi = u^2, \quad \lim_{|x|\to\infty}\phi(x) = 0 \end{cases} \tag{4}$$

  where
  $|u|^2 : \mathbb{R}^3 \to \mathbb{R}$: particle density
  $\phi : \mathbb{R}^3 \to \mathbb{R}$: electric potential
  $\lambda \in \mathbb{R}$: coupling constant

- This system describes systems of identically charged particles interacting each other in the case that magnetic effects could be ignored, and its solution is, in particular, a standing wave for such a system.

Reducing to a single equation

- One can solve ϕ in term of $u \in H^1(\mathbb{R}^3)$, i.e.,

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|^2}\,dy \in D^{1,2}(\mathbb{R}^3) \tag{5}$$

where $D^{1,2}(\mathbb{R}^3) = \{v \in L^6(\mathbb{R}^3) | \nabla v \in L^2(\mathbb{R}^3)\}$.

- Define an energy functional

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\,dydx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,dx \tag{6}$$

where $u \in H^1(\mathbb{R}^3), \|u\|^2 = \int_{\mathbb{R}^3} |\nabla u|^2 + u^2\,dx$.

- Then, a critical point $u \in H^1(\mathbb{R}^3)$ is a solution of

$$-\Delta u + u + \lambda\phi_u u = |u|^{p-2}u. \tag{7}$$

Known results (when $\lambda > 0$)

- If $3 < p < 6$, there is a positive radial solution for all $\lambda > 0$ (Ruiz, 2006).

  If $2 < p \leq 3$, there is no solution for $\lambda \geq 1/4$ and there are at least two solutions for sufficiently small $\lambda > 0$ (Ruiz, 2006).

- There is a positive least energy solution for $3 < p < 6$ and all $\lambda > 0$ (Azzollini and Pomponio, 2008).

- There are infinitely many radial solutions for $3 < p < 6$ and all $\lambda > 0$ (Ambrosetti and Ruiz, 2008).

Questions (when $\lambda > 0$)

- **Question 1** Is any positive solution radially symmetric up to translation? $\leftarrow$ Not known
- **Question 2** Are there sign changing solutions?
  $\leftarrow$ YES for $4 < p < 6$ and every $\lambda > 0$

Consider a problem

$$\begin{cases} -\Delta u + u + \phi u = f(u), \\ -\Delta \phi = u^2, \quad \lim_{|x|\to\infty} \phi(x) = 0 \end{cases} \tag{8}$$

We remind that if $f(u) = |u|u$, there is no solution but if $f(u) = |u|^{p-1}u$, $2 < p < 6$, there are infinitely many solution.

- **Question 3** If $f(u) = |u|u \log |u|$, is there a solution? $\leftarrow$ YES

Answers to Questions 3 and 4

Theorem (Kim and S, 2011)

For $4 < p < 6$, Choose an arbitrary natural number k. then there exists a solution of NSP system changing sign exactly k-times.

Theorem (S, 2011)

For the nonlinearity $f(u) = |u|u \log |u|$, there exist infinitely many solutions of NSP system.

Idea of proof of the first theorem (1)

We introduce some notations. Fix $k \in \mathbb{N}$.

- $\Lambda := \{r = (r_1, \cdots, r_k) \mid 0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := \infty\}$
- $A_{i,r} := \{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_i\}$ for $i = 1, \cdots, k+1$ and $r \in \Lambda$
- $$H_{i,r} := \{u \in L^2(A_{i,r}) : |\nabla u| \in L^2(A_{i,r}), u(x) = u(|x|), \\ u = 0 \text{ on } \partial A_{i,r}\}$$
- $\widetilde{H}_r = H_{1,r} \times \cdots \times H_{k+1,r}$
- $\|u\|_{i,r} := \int_{A_{i,r}} (u^2 + |\nabla u|^2)$

Idea of proof of the first theorem (2)

- Define an energy functional $E_r : \widetilde{H} \to \mathbb{R}$ by

$$E_r(u_1, \cdots, u_{k+1}) = \frac{1}{2}\sum_{i=1}^{k+1} \|u_i\|_{i,r}^2 + \frac{1}{4}\sum_{i,j}\int_{A_{i,r}}\int_{A_{i,r}} \frac{u_i^2(x)u_i^2(y)}{|x-y|}dydx - \frac{1}{p}\sum_{i=1}^{k+1}\int_{A_{i,r}} |u_i|^p \quad (9)$$

where $u_i \in H_{i,r}$ for $i = 1, \cdots, k+1$.

- Each component u_i of a critical point $(u_1, \cdots, u_{k+1})$ of E_r satisfies

$$\begin{cases} -\Delta u_i + u_i + \phi u_i = |u_i|^{p-2}u_i \text{ in } A_{i,r}, \\ -\Delta \phi = \left(\sum_{i=1}^{k+1} u_i\right)^2, \quad \lim_{|x|\to\infty} \phi(x) = 0. \end{cases} \quad (10)$$

Idea of proof of the first theorem (3)

- Define a set

$$\mathcal{N}_r = \{(u_1, \cdots, u_{k+1}) \in \widetilde{H}_r \mid u_i \neq 0, \partial_{u_i} E_r(u_1, \cdots, u_{k+1}) u_i = 0 \text{ for } \forall i\}$$

- Consider a constrained minimization problem

$$W_r := \min_{u \in \mathcal{N}_r} E_r(u)$$

- Show that W_r is attained by a minimizer $w_r \in \mathcal{N}_r$ and every minimizer is a critical point of E_r.

- Redefine w_r as $(|w_1|, -|w_2|, \ldots, (-1)^{i+1}|w_i|, \ldots, (-1)^{k+2}|w_{k+1}|)$ Then it is still a minimizer.

Idea of proof of the first theorem (4)

- Minimize $E_r(w_r)$ over all $r \in \Lambda$. Show that there is a minimizer r_0.
- Finally, show that w_{r_0} solves the problem in whole domain $\mathbb{R}^3$. In other words, it solves the problem at $\partial A_{i,r_0}$ for all i.

Here are some comments.

- We need the restriction of the range of $p \in (4,6)$ to attain W_r for each $r \in \Lambda$.
- For case of remaining range of $p \in (3,4)$, problem is still open.

Second theorem

In fact, we can prove the following more general result.

Theorem (S,2011)

Suppose the following structure conditions hold

- $(F1)$ f *is continuous and odd;*
- $(F2)$ $\lim_{t\to 0} f(t)/t = 0, \limsup_{t\to\infty} f(t)/t^p < \infty$ *for some* $p \in (1,5)$*;*
- $(F3)$ $\frac{2f(t)}{t^2} - \frac{F(t)}{t^3}$ *increase to infinity as* $t \to \infty$*.*

Then there exist infinitely many radial solutions of (8)*.*

Note that $f(t) = |t|t\log|t|$ satisfies $(F1) \sim (F3)$.

Known results (when $\lambda < 0$)

Consider the nonlinear Schrödinger-Poisson equation with negative λ:

$$-\Delta u + u + \lambda \phi_u u = f(u). \tag{11}$$

- If $f(u) = -|u|^{p-2}u$, $2 < p < 4$, there is a positive radial solution for all $\lambda < 0$ (Mugnai, 2011).
- If $f(u) = -|u|^{p-2}u$, $4 \leq p < 6$, there is a positive radial solution for countably many $\lambda < 0$ (Mugnai, 2011).

Questions (when $\lambda < 0$)

- **Question 4** For $4 \leq p < 6$ and $f(u) = -|u|^{p-2}u$, can we widen the existence range of $\lambda < 0$?

  $\leftarrow$ YES (for sufficiently large $|\lambda|$, there is a solution)

- **Question 5** For the nonlinearity $|u|^{p-2}u$, is there a solution?

  $\leftarrow$ YES (for sufficiently large $|\lambda|$, there is a solution)

Answers for Question 4 and 5

Theorem (Jeong and S, 2012)

Suppose that $\lambda < 0$ and f satisfies

(F1) $f : \mathbb{R} \to \mathbb{R}$ *is continuous.*

(F2) $\lim_{s\to 0} f(s)/s = 0$ *and* $\limsup_{|s|\to\infty} |f(s)|/|s|^p < \infty$ *for some* $p \in (1, 5)$.

Then, for sufficiently large $|\lambda|$, there exists a solution.

Idea of proof (1)

- By defining $u(x) = \varepsilon v(x)$ with $\varepsilon = 1/\sqrt{-\lambda}$, the equation is equivalent to

$$-\Delta v + v - \phi_v v = f_\varepsilon(v) \qquad \text{in } \mathbb{R}^3, \tag{12}$$

where

$$f_\varepsilon(v) = \frac{1}{\varepsilon} f(\varepsilon v)$$

and we easily see that $f_\varepsilon(v) \to 0$ as $\varepsilon \to 0$ from (F2).

- As $\varepsilon \to 0$, we obtain an equation

$$-\Delta v + v - \phi_v v = 0 \qquad \text{in } \mathbb{R}^3,$$

which is called the Choquard equation.

Idea of proof (2)

- Define a functional $I_\varepsilon(u) = I_0(u) + J_\varepsilon(u) on H := H^1_r(\mathbb{R}^3)$ by

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\, dx, \quad J_\varepsilon(u) = \int_{\mathbb{R}^3} F_\varepsilon(u)\, dx,$$

  where $F_\varepsilon(u) = \frac{1}{\varepsilon^2} F(\varepsilon u)$.

- A critical point of I_ε is a solution of our problem.

- We want to find a critical point of I_ε for sufficiently small $\varepsilon > 0$, i.e., for sufficiently large $|\lambda|$.

- Since $J_\varepsilon(u) \to 0$ as $\varepsilon \to 0$, I_ε is a small perturbation of I_0 for small $\varepsilon > 0$.

Idea of proof (3)

(M1) $I_0(0) = 0$, there exist $c, r > 0$ such that if $\|u\| = r$, then $I_0(u) \geq c$ and there exists a $v_0 \in H$ such that $\|v_0\| > r$ and $I_0(v_0) < 0$;

(M2) there exists a critical point $u_0 \in H$ of I_0 such that

$$I_0(u_0) = C_0 := \min_{\gamma \in \Gamma} \max_{s \in [0,1]} I_0(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0,1], H) \mid \gamma(0) = 0,\ \gamma(1) = v_0\}$;

(M3) it holds that

$$C_0 = \inf_{\{u \in H \mid I_0'(u) = 0, u \neq 0\}} I_0(u);$$

(M4) the set $S := \{u \in H \mid I_0'(u) = 0,\ I_0(u) = C_0\}$ is compact in H;

(M5) there exists a curve $\gamma_0(s) \in \Gamma$ passing through u_0 at $s = s_0$ and satisfying

$$I_0(u_0) > I_0(\gamma_0(s)) \quad \text{for all } s \neq s_0.$$

(J) J_ε and J_ε' are compact and satisfy for any $M > 0$,

$$\lim_{\varepsilon \to 0} \sup_{\|u\| \leq M} |J_\varepsilon(u)| = \lim_{\varepsilon \to 0} \sup_{\|u\| \leq M} \|J_\varepsilon'(u)\| = 0;$$

Idea of proof (4)

- We define a modified mountain pass energy level of I_ε

$$C_\varepsilon = \min_{\gamma \in \Gamma_M} \max_{s \in [0,1]} I_\varepsilon(\gamma(s)),$$

where

$$\Gamma_M = \left\{ \gamma \in \Gamma \mid \sup_{s \in [0,1]} \|\gamma(s)\| \le M \right\}$$

$$M := 2 \max \left\{ \sup_{u \in S} \|u\|, \sup_{s \in [0,1]} \|\gamma_0(s)\| \right\}.$$

- By the choice of M, we see that $\gamma_0 \in \Gamma_M$ and thus

$$C_0 = \min_{\gamma \in \Gamma_M} \max_{s \in [0,1]} I_0(\gamma(s)).$$

Idea of proof (5)

- We can prove that $\lim_{\varepsilon \to 0} C_\varepsilon = C_0$.
- Assume that there is no critical point of I_ε on any small neighborhood of S. Then, we can deform $\gamma_0(s)$ along the direction of $-I'_\varepsilon$ and obtain a curve $\tilde{\gamma}(s) \in \Gamma_M$ satisfying

$$\max_{s \in [0,1]} I_\varepsilon(\tilde{\gamma}(s)) \le \max_{s \in [0,1]} I_\varepsilon(\gamma_0(s)) - \delta = C_0 - \delta,$$

  where $\delta > 0$ is a constant independent of $\varepsilon > 0$.
- Then, for small $\varepsilon > 0$, we see $\max_{s \in [0,1]} I_\varepsilon(\tilde{\gamma}(s)) < C_\varepsilon$, which is a contradiction.

Thank you for your attention!